Exam

Electricity and Magnetism 1

Thursday July 11, 2013
9:00-12:00

Place your student card at the right side of the table.

Write your name and student number on *every* sheet.
***Write clearly*.**
Use a *separate* sheet for each question.

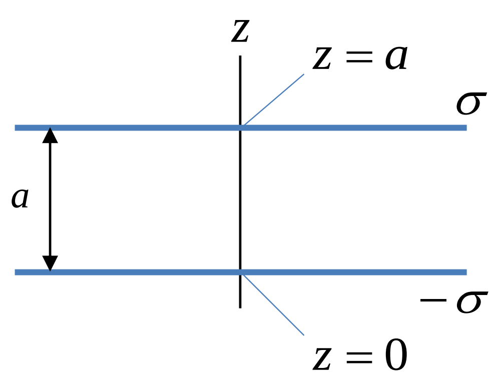
This exam consists of 4 questions.
All questions are of equal weight.

PROBLEM 1

Score: a+b+c+d+e+f =2+4+3+2+3+4=18

a) Write down Gauss's law for the electric field in integral form.
b) Use this integral form to derive Gauss's law in differential form.

The plates of a parallel plate capacitor uniformly charged with surface charge σ (top plate) and $-\sigma$ (bottom plate). The distance between the plates is a and the surface area of each plate is S. All edge effects may be neglected.



c) Show that $\vec{E} = 0$ when $z > a$ and when $z < 0$ and $\vec{E} = -\frac{\sigma}{\varepsilon_0}\hat{z}$ when $0 < z < a$.
d) Find the potential difference $\Delta V = V_+ - V_-$ between the positive and negative plate and show that the surface charge density σ is $\sigma = \frac{\Delta V \varepsilon_0}{a}$.
e) Find the capacity C of the capacitor.

The space between the plates is completely filled with a linear dielectric with:

$$\varepsilon(z) = \frac{\varepsilon_0}{\left(1 - \frac{z}{2a}\right)}$$

During filling the potential between the plates is kept constant (at ΔV).

f) Show that:

$$\acute{\sigma} = \frac{4}{3}\sigma$$

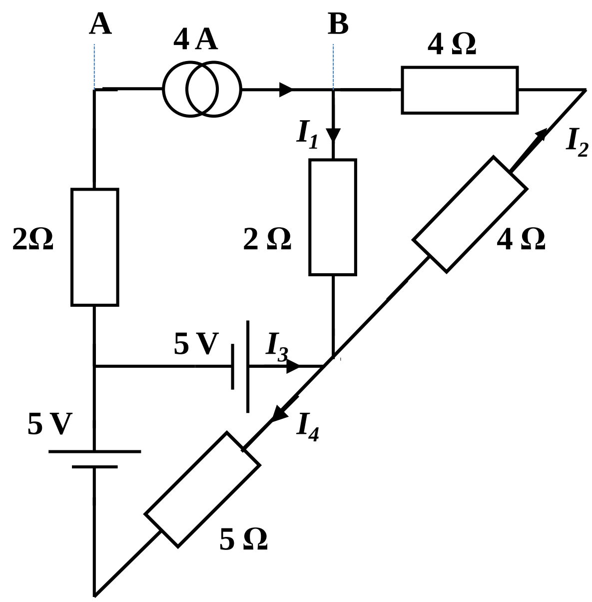
with $\acute{\sigma}$ the surface charge density on the top plate in the situation with the linear dielectric.

PROBLEM 2

Score: a+b+c+d+e+f=3+3+3+4+3+2=18

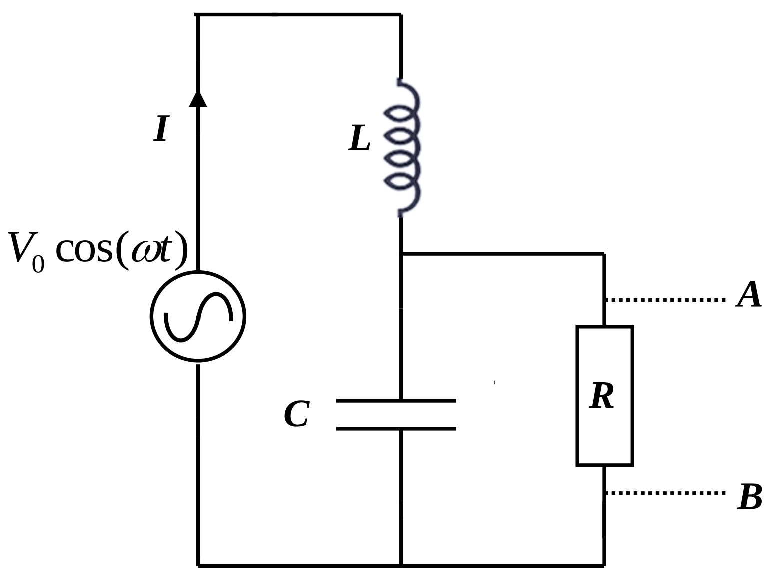
Consider the electric circuit in the side figure.

a) Find all the node equations (Kirchhoff 1). Show that one of this equations can be derived from the others.
b) Find all the loop equations (Kirchhoff 2).
c) Find I_1, I_2, I_3, and I_4 and the potential difference $V_{AB} = V_B - V_A$ over the current source.



Consider the electric circuit in the figure below. The stationary voltage source is described (in the real representation) by $V = V_0 \cos(\omega t)$.

d) Find the potential difference $V_{AB} = V_B - V_A$ over the resistor in the complex representation.
e) Find the real potential difference $V_{AB} = V_B - V_A$ over the resistor.
f) At what value of the frequency ω is the amplitude of V_{AB} at its maximum?



PROBLEM 3

Score: a+b+c+d+e+f=3+4+2+4+3+2=18

General: *Use cylinder coordinates. All edge effects may be neglected.*

Consider an infinity long wire along the *z*-axis, the wire carries a current $\vec{I}_1 = I_1\hat{z}$ in the positive *z*-direction (see left figure).

a) Find the magnetic field $\vec{B}$ everywhere.

We place a long hollow cylinder coaxially around the wire (see right figure). The inner- and outer radius of the cylinder are a and b, respectively. The hollow cylinder carries a current in the negative *z*-direction, this current is described by the following volume current density:

$$\vec{J}(s) = -J_0\frac{s^2}{a^2}\hat{z}; a \leq s \leq b$$

b) Proof that the total current $\vec{I}_2$ in the hollow cylinder due to this charge density is:

$$\vec{I}_2 = -I_2\hat{z} \text{ met } I_2 = \frac{1}{2}\pi a^2 J_0\left(\left(\frac{b}{a}\right)^4 - 1\right)$$

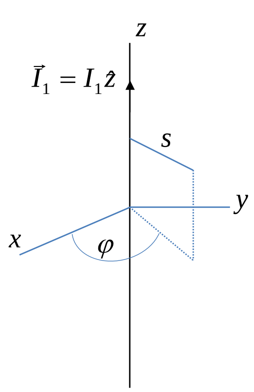
c) What is the dimension of I_2 and of J_0?.

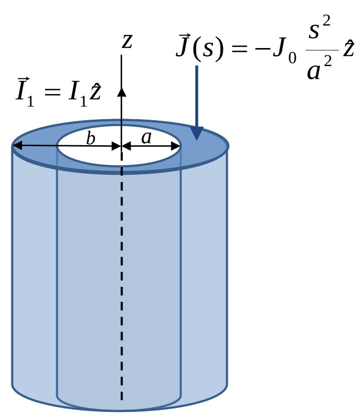
d) Find the magnetic field in the three regions $s < a$ and $a \leq s \leq b$ and $s > b$.

The region with $s < a$ is completely filled with a paramagnetic material with magnetic susceptibility χ_m.

e) Find the magnetic field $\vec{B}$ in the region $s < a$.

f) Find the bound surface current on the surface of the paramagnetic material at $s = a$.

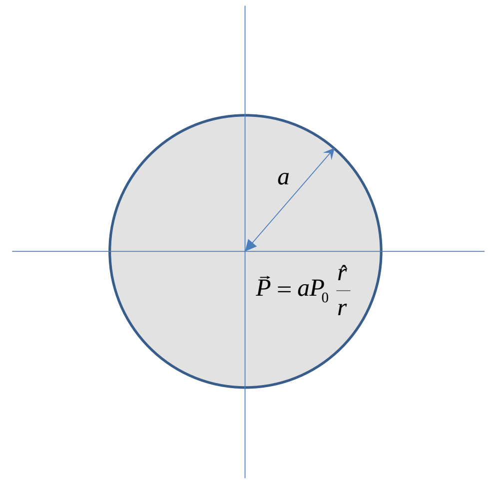




PROBLEM 4

Score: a+b+c+d+e =2+2+2+6+6=18

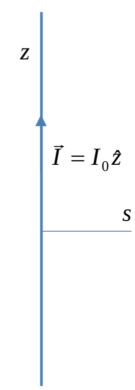
Consider a solid sphere (of radius a) with its centre at the origin (see figure). The sphere carries a fixed polarization $\vec{P} = aP_0 \frac{\hat{r}}{r}$.



a) Find the bound surface charge density σ_b at the surface of the sphere.
b) Find the bound volume charge density ρ_b in the sphere.
c) Show that the sphere is neutral.

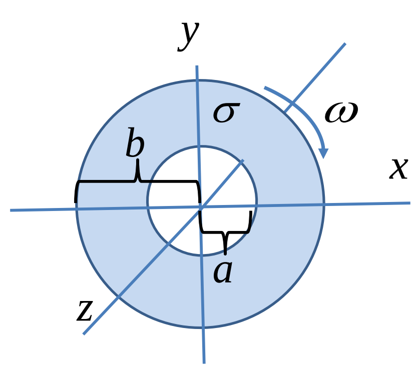
A infinitely long wire lies along the z-axis and carries a current $\vec{I} = I_0\hat{z}$ (see figure).

Find the vector potential $\vec{A}$ at a distance s to the wire. You may use your knowledge of the magnetic field of an infinite wire. Choose your own reference point at which the vector potential is zero.



Consider a disk with surface charge density σ, inner radius a and outer radius b (see figure). The disk rotates clockwise around the z-axis with angular velocity ω.

d) Find the magnetic dipole moment $\vec{m}$ of this disk.



Solutions

PROBLEM 1

a)

Gauss's law in integral form $\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0}$

b)

$$\oint \vec{E} \cdot d\vec{a} = \int (\vec{\nabla} \cdot \vec{E})\, d\tau = \frac{Q_{enc}}{\varepsilon_0} = \int \frac{\rho}{\varepsilon_0}\, d\tau$$

In the second step the divergence theorem is used.

It follows

$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$,

which is Gauss's law in differential form.

c)

The situation has infinite plate symmetry thus all fields are in the *z*-direction. First the electric field of one plate. Use a Gaussian pillbox as on pages 74-75 of Griffiths. Apply Gauss's law: $\oint \vec{E} \cdot d\vec{a} = AE + AE = \frac{\sigma A}{\varepsilon_0}$, with the surface A parallel to the plate. For the positive plate we find,

$\vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{z}$ if $z > a$ and $\vec{E} = \frac{-\sigma}{2\varepsilon_0}\hat{z}$ if $z < a$, and for the negative plate $\vec{E} = \frac{-\sigma}{2\varepsilon_0}\hat{z}$ if $z > 0$ and $\vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{z}$ if $z < 0$. Superposition shows that no field exists outside the capacitor and that the field inside the capacitor is: $\vec{E} = -\frac{\sigma}{\varepsilon_0}\hat{z}$

d)

$$\Delta V = V_+ - V_- = -\int_-^+ \vec{E} \cdot d\vec{l} = -\int_0^a \left(-\frac{\sigma}{\varepsilon_0}\right) dz = \frac{\sigma a}{\varepsilon_0}$$

and

$$\sigma = \frac{\Delta V \varepsilon_0}{a}$$

e)

$$C = \frac{Q}{\Delta V} = \frac{\sigma S}{\Delta V} = \frac{\varepsilon_0 S}{a}$$

f)

The procedure is more or less identical to the solution to c) but now using Gauss's law for the $\vec{D}$-field:

$$\oint \vec{D} \cdot d\vec{a} = Q_{free}^{enc}$$

We find the $\vec{D}$-field inside the capacitor

$$\vec{D} = -\acute{\sigma}\hat{z}$$

The $\vec{E}$-field follows from

$$\vec{E} = \frac{\vec{D}}{\varepsilon} = -\frac{\acute{\sigma}}{\varepsilon_0}\left(1 - \frac{z}{2a}\right)\hat{z}$$

And the potential difference ΔV from:

$$\Delta V = V_+ - V_- = -\int_-^+ \vec{E} \cdot d\vec{l} = -\int_0^a \left(-\frac{\acute{\sigma}}{\varepsilon_0}\left(1 - \frac{z}{2a}\right)\right) dz = \frac{3}{4}\frac{\acute{\sigma} a}{\varepsilon_0}$$

and

$$\acute{\sigma} = \frac{4}{3}\frac{\Delta V \varepsilon_0}{a} = \frac{4}{3}\sigma$$

PROBLEM 2

a)
We have three node equations. From Kirchhoff 1,

K1: $4 - I_1 + I_2 = 0$
K2: $I_1 + I_3 - I_2 - I_4 = 0$
K3: $I_4 - I_3 - 4 = 0$

These equations are not independent e.g.: –(K1+K3)=K2.

b)
Leaving out the current source we have two real loops, from Kirchhoff 2,

M1 (clockwise): $2I_1 + 4I_2 + 4I_2 = 0$
M2 (clockwise): $5 + 5 - 5I_4 = 0$

c)
From M2: $I_4 = 2$ A. Use K3: $I_3 = -2$ A. From M1: $I_1 = -4I_2$; combine with K1 and find: $I_2 = I_1 - 4 = -4I_2 - 4$ and $I_2 = -\frac{4}{5}$ A. Use K1: $I_1 = \frac{16}{5}$ A.

The potential difference over the current source results from Kirchhoff 2 for the loop with the current source (clockwise):

$$-8 + V_{AB} - 2I_1 - 5 = 0 \Longrightarrow V_{AB} = 13 - 2I_1 = 13 - \frac{32}{5} = 19\frac{2}{5}V$$

d)
Define currents I (upwards through the voltage source), I_1 (downward through the capacitor) and I_2 (downward through the resistor).
The potential difference is: $V_{AB} = V_B - V_A = -RI_2$

Kirchhoff 1 (1 independent node equation),

$I = I_1 + I_2$

Kirchhoff 2 (2 loops, clockwise),

M1: $V_0 - Z_L I - Z_C I_1 = 0$
M2: $Z_C I_1 - R I_2 = 0$

Use the second loop equation:

$$I_1 = \frac{R I_2}{Z_C}$$

And substitute this together with the node equation in the first loop equation,

$V_0 - Z_L(I_1 + I_2) - Z_C \frac{R I_2}{Z_C} = 0 \Rightarrow V_0 - Z_L(\frac{R I_2}{Z_C} + I_2) - R I_2 = 0$ and

$$I_2 = \frac{V_0}{Z_L + R + \frac{R Z_L}{Z_C}}$$

This results in,

$$V_{AB} = -R I_2 = -\frac{R V_0}{Z_L + R + \frac{R Z_L}{Z_C}}$$

With: $Z_L = i\omega L$ en $Z_C = \frac{-i}{\omega C}$ we find V_{AB} in the complex representation,

$$V_{AB} = -\frac{R V_0}{i\omega L + R + \frac{R i \omega L}{\frac{-i}{\omega C}}} = \frac{-V_0}{(1 - \omega^2 LC) + i\frac{\omega L}{R}}$$

e)

Converting to the real representation,

$$|V_{AB}| = \left|\frac{-V_0}{(1 - \omega^2 LC) + i\frac{\omega L}{R}}\right| = \frac{V_0}{\sqrt{(1 - \omega^2 LC)^2 + \left(\frac{\omega L}{R}\right)^2}}$$

and

$$\arg(V_{AB}) = \arg(-V_0) - \arg\left((1 - \omega^2 LC) + i\frac{\omega L}{R}\right) = \pi - \tan^{-1}\left(\frac{\omega L}{R(1 - \omega^2 LC)}\right)$$

The real potential difference becomes,

$$V_{AB} = \frac{V_0}{\sqrt{(1-\omega^2 LC)^2 + \left(\frac{\omega L}{R}\right)^2}} \cos(\omega t + \varphi)$$

with $\varphi = \pi - \tan^{-1}\left(\frac{\omega L}{R(1-\omega^2 LC)}\right)$

f)

Amplitude is at maximum if $f(\omega) = (1-\omega^2 LC)^2 + \left(\frac{\omega L}{R}\right)^2$ is at minimum.

The extreme values of $f(\omega)$ are found with $\frac{\partial f}{\partial \omega} = 0$. Differentiating,

$$2(1-\omega^2 LC)(-2\omega LC) + 2\omega\left(\frac{L}{R}\right)^2 \Rightarrow \omega = 0 \vee \omega^2 = \frac{1}{LC}\left(1 - \frac{1}{2}\frac{L}{R^2 C}\right)$$

There are two non-zero solutions of which only the positive solution ω_+ is physical.

$$\omega_+ = \sqrt{\frac{1}{LC}\left(1 - \frac{1}{2}\frac{L}{R^2 C}\right)}$$

Which of the solutions ($\omega = 0$ of $\omega = \omega_+$) results in the maximum amplitude depends on the values of R, L, and C.

From substitution of ω_+ in the expression for the amplitude we deduce that if

$$\frac{L}{R^2 C}\left(1 - \frac{1}{4}\frac{L}{R^2 C}\right) - 1 < 0$$

then $\omega = \omega_+$ leads to the maximum amplitude, else $\omega = 0$.

However, the inequality above is satisfied for all values of R, L, en C, this follows from,

$$x\left(1 - \frac{1}{4}x\right) - 1 = -(x^2 - 4x + 4) = -(x-2)^2.$$

Thus $\omega = \omega_+$ results in the maximum amplitude. If $\frac{L}{R^2 C} = 2$ then $\omega_+ = 0$

PROBLEM 3

a)

This is the symmetry of the long wire, the field is in the $\hat{\varphi}$-direction. Use Ampere's law with a circle of radius s around the wire.

$$\oint \vec{B} \cdot d\vec{l} = 2\pi s B = \mu_0 I_{enc} \Rightarrow B = \frac{\mu_0 I_1}{2\pi s}$$

and

$$\vec{B} = \frac{\mu_0 I_1}{2\pi s} \hat{\varphi}$$

b)

$$I_2 = \int \vec{J} \cdot d\vec{a} = \int_a^b J(s) 2\pi s ds = \int_a^a J_0 \frac{s^2}{a^2} 2\pi s ds = \frac{2\pi J_0}{a^2} \int_a^b s^3\, ds = \frac{\pi J_0}{2a^2}(b^4 - a^4)$$

and

$$\vec{I}_2 = -I_2 \hat{z} \;\text{ met } I_2 = \frac{1}{2}\pi a^2 J_0 \left(\left(\frac{b}{a}\right)^4 - 1\right)$$

c)

The dimension of I_2 are Ampere=Coulomb per second.

The volume current density J_0 has dimension Ampere per unit area=Coulomb/m^{2}s

d)

Use Ampere's law with a circle of radius s. In case $s < a$ the only enclosed current is $\vec{I}_1$ through the wire. Consequently in this region the magnetic field is identical as under a)

$$\vec{B} = \frac{\mu_0 I_1}{2\pi s} \hat{\varphi}$$

In region $a \le s \le b$ the enclosed current is $\vec{I}_1$ of the wire plus a part $I_2(s)$ contributed by the volume current. This part depends on the radial coordinate s,

$$I_2(s) = \int_a^s J(\acute{s}) 2\pi \acute{s} d\acute{s} = \int_a^s J_0 \frac{\acute{s}^2}{a^2} 2\pi \acute{s} d\acute{s} = \frac{\pi J_0}{2a^2}(s^4 - a^4)$$

The total current enclosed is,

$$\vec{I}_{enc} = I_1\hat{z} - I_2(s)\hat{z} = \left(I_1 - \frac{1}{2}\pi a^2 J_0\left(\left(\frac{s}{a}\right)^4 - 1\right)\right)\hat{z}$$

And with Ampere´s law we find,

$$\vec{B} = \frac{\mu_0\left(I_1 - \frac{1}{2}\pi a^2 J_0\left(\left(\frac{s}{a}\right)^4 - 1\right)\right)}{2\pi s}\hat{\varphi}$$

If $s > b$ the total current enclosed is the sum of the currents through the wire and the hollow cylinder and it follows,

$$\vec{B} = \frac{\mu_0\left(I_1 - \frac{1}{2}\pi a^2 J_0\left(\left(\frac{b}{a}\right)^4 - 1\right)\right)}{2\pi s}\hat{\varphi} = \frac{\mu_0(I_1 - I_2)}{2\pi s}\hat{\varphi}$$

e)

Use Ampere´s law for the $\vec{H}$-veld:

$$\oint \vec{H}\cdot d\vec{l} = 2\pi s H = I_{free}^{enc} = I_1$$

and $\vec{H} = \frac{I_1}{2\pi s}\hat{\varphi}$

The magnetic field $\vec{B}$ follows from:

$$\vec{B} = \mu\vec{H} = \mu_0(1+\chi_m)\vec{H} = \mu_0(1+\chi_m)\frac{I_1}{2\pi s}\hat{\varphi}$$

f)

$$\vec{K}_b(s=a) = \vec{M} \times \hat{n}\big|_{s=a} = \chi_m \vec{H} \times \hat{n}\big|_{s=a} = \frac{\chi_m I_1}{2\pi s}\hat{\varphi} \times (-\hat{s})|_{s=a} = \frac{\chi_m I_1}{2\pi a}\hat{z}$$

PROBLEM 4

a)

$$\sigma_b = \vec{P} \cdot \hat{n}\big|_{r=a} = aP_0 \frac{\hat{r}}{r} \cdot \hat{r}\Big|_{r=a} = P_0$$

b)

In spherical coordinates:

$$\rho_b = -\vec{\nabla} \cdot \vec{P} = -\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \left(aP_0 \frac{1}{r}\right) = -\frac{aP_0}{r^2}$$

c)

Total bound charge at the surface of the sphere: $4\pi a^2 P_0$

Total bound volume charge in the sphere:

$$\int_0^a -\frac{aP_0}{r^2} 4\pi r^2 dr = -4\pi a^2 P_0$$

d)

Vector potential is always in the direction of the current, this follows from,

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{I}}{r} d\acute{l}$$

So $\vec{A}$ is in the $\hat{z}$-direction.

The magnetic field $\vec{B}$ at distance s from the wire is,

$\vec{B} = \frac{\mu_0 I_0}{2\pi s}\hat{\varphi}$

We have:

$\vec{B} = \vec{\nabla} \times \vec{A}$ and consequently, $B_\varphi = \frac{\mu_0 I}{2\pi s} = \left(\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s}\right) = 0 - \frac{\partial A_z}{\partial s}$ this leads to:

$\frac{\partial A_z}{\partial s} = -\frac{\mu_0 I_0}{2\pi s} \Longrightarrow A_z = -\frac{\mu_0 I_0}{2\pi}\ln(s) + \text{constant}$

We are free to choose the constant. If we choose the constant equal to $\frac{\mu_0 I_0}{2\pi}\ln(a)$, with a an arbitrary distance to the wire then.

$A_z = -\frac{\mu_0 I_0}{2\pi}\ln\left(\frac{s}{a}\right)$

and $A_z(s = a) = 0$

Finally: $\vec{A} = -\frac{\mu_0 I_0}{2\pi}\ln\left(\frac{s}{a}\right)\vec{z}$

e)
Assume the disk to be a superposition of circular wires with radius r $(a < r < b)$ and thickness dr. Each of these wire will carry a current$I(r) = \sigma\omega r dr$ and the magnitude of the magnetic dipole moment dm of such a wire becomes:

$$dm = \pi r^2 I(r) = \pi r^2 \sigma\omega r dr = \pi\sigma\omega r^3 dr$$

For the complete disk,

$$m = \pi\sigma\omega \int_a^b r^3 dr = \frac{1}{4}\pi\sigma\omega(b^4 - a^4)$$

The direction of the magnetic dipole moment is the $-\hat{z}$-direction (right hand rule). Thus,

$\vec{m} = -\frac{1}{4}\sigma\omega\pi(b^4 - a^4)\hat{z}$